SOME RESULTS IN THE LOCATION OF THE ZEROS OF LINEAR COMBINATIONS OF POLYNOMIALS

BY

ZALMAN RUBINSTEIN[1]

We study here the location of the zeros of linear combinations of polynomials of the form $f(z) - \lambda g(z)$, where $f(z)$ and $g(z)$ are arbitrary polynomials with complex coefficients and λ is a complex number. It is known [3] that this question is closely connected with the study of the zeros of polynomials of the form $(z-\alpha)^n - \lambda(z-\beta)^r$, which indeed is the main object of this paper.

We start with a particular case.

THEOREM 1. *Let the polynomials* $f(z) = z^n + \cdots$, *and* $g(z) = z^r + \cdots$, $n = 2r$, *have zeros in the circles* $|z - a| \leqq r_1$ *and* $|z - b| \leqq r_2$, *respectively, then all the zeros of the polynomial*

$$f(z) - \lambda g(z) \tag{1}$$

are in the union of the n circles

$$\left| z - a - \frac{1}{2}\lambda^{2/n} + \lambda^{1/n}\left(a - b + \frac{1}{4}\lambda^{2/n}\right)^{1/2} \right| \leqq (r_1 + r_2)^{1/2}|\lambda|^{1/n} + r_1, \tag{2}$$

where $\lambda^{1/n}$ *assumes all the nth roots of* λ.

Proof. The equation $f(z) - \lambda g(z) = 0$ can be replaced by Grace's theorem [3] by the equation $(z-\alpha)^n - \lambda(z-\beta)^{n/2} = 0$, where $|\alpha - a| \leqq r_1$, and $|\beta - b| \leqq r_2$.

Solving for z we obtain

$$z = \alpha + \frac{1}{2}\lambda^{2/n} \pm \lambda^{1/n}\left[(\alpha - \beta) + \frac{1}{4}\lambda^{2/n}\right]^{1/2}.$$

Denoting generically the region $|z - c| \leqq R$ by $C(c, R)$ we have

$$\alpha - \beta \in C(a - b, r_1 + r_2),$$

$$\left(\alpha - \beta + \frac{1}{4}\lambda^{2/n}\right)^{1/2} \in C\left(\pm\left(a - b + \frac{1}{4}\lambda^{2/n}\right),\ (r_1 + r_2)^{1/2}\right);$$

hence

Received by the editors August 21, 1963 and, in revised form, March 11, 1964.

(1) This work was supported by the Air Force Office of Scientific Research.

$$z \in C\left(a + \frac{1}{2}\lambda^{2/n} \pm \lambda^{1/n}\left(a - b + \frac{1}{4}\lambda^{2/n}\right)^{1/2}, (r_1 + r_2)^{1/2}|\lambda|^{2/n} + r_1\right).$$

(2) follows since, by assumption, n is an even number.

The result is sharp for $\lambda = 0$, and for $a = b$.

For the general case we have

THEOREM 2[2]. *Let $f(z) = z^n + \cdots, g(z) = z^r + \cdots, n > r$, have zeros in the circles $|z - a| \leqq r_1$ and $|z - b| \leqq r_2$, respectively. Then all the zeros of the polynomial $f(z) - \lambda g(z)$ are in the circle*

$$|z - a| \leqq r_1 + d,$$

where d is the positive root of the equation

$$d^{n/r} - Md - N = 0 \tag{3}$$

with

$$M = |\lambda|^{1/r}, \qquad N = |\lambda|^{1/r}(|a - b| + r_1 + r_2).$$

Proof. Consider the equation

$$(z - \alpha)^n = \lambda(z - \beta)^r, \qquad |a - \alpha| \leqq r_1, \qquad |b - \beta| \leqq r_2.$$

For z_0 satisfying $(z_0 - \alpha)^n = \lambda(z_0 - \beta)^r$, $(z_0 - \alpha)^{n/r-1} = \lambda^{1/r}((z_0 - \beta)/(z_0 - \alpha))$. Let d_1 be a positive number satisfying

$$d_1^{n/r} - Md_1 - N > 0.$$

For $|z_0 - \alpha| \geqq d_1$, $(z_0 - \beta)/(z_0 - \alpha)$ belongs to the circle $|z - 1| \leqq |\alpha - \beta|/d_1$; hence

$$\left|\lambda^{1/r}\frac{z_0 - \beta}{z_0 - \alpha}\right| \leqq |\lambda|^{1/r}\left(1 + \frac{|\alpha - \beta|}{d_1}\right),$$

but

$$|z_0 - \alpha|^{n/r-1} \geqq d_1^{n/r-1} > |\lambda|^{1/r}\left(1 + \frac{|\alpha - \beta|}{d_1}\right),$$

for all α, β such that $|\alpha - a| \leqq r_1$, and $|\beta - b| \leqq r_2$. We get a contradiction, which proves that $|z_0 - \alpha| < d_1$.

It is worthwhile to remark that if $M + N > 1$ an estimate for the positive zero d is the expression

$$\frac{(n - r)(M + N)^{n/n-r} + rN}{(n - r)(M + N) + rN} \leqq (M + N)^{r/n-r}.$$

For $M + N < 1$ a bound for the same is $((n - r + rN)/(n - rM)) \leqq 1$.

[2] Theorem 2 was proved independently and by a different method by Mishael Zedek [5].

Different estimates can be obtained by means of estimates similar to those used in the proof of Theorem 2, which are sharp for $\lambda = 0$ or asymptotically for $\lambda \to \infty$. We indicate some of them which are of a relatively simple form.

THEOREM 3. *Let $f(z)$ and $g(z)$ be as in Theorem 2. All the zeros of the polynomial $f(z) - \lambda g(z)$ are in each of the following regions*:

$$|z| \leqq \frac{|a| - r_1}{d(|a| - r_1) - 1}[(|b| + r_2)d + 1], \tag{4}$$

where $r > n$, $d = |\lambda|^{1/r}(r_1 + |a|)^{-n/r}$, and $d(|a| - r_1) - 1 > 0$.

$$|z - b| \leqq r_2 + 2\,\mathrm{Max}[|\lambda|^{-(1/r-n)}, (|a - b| + r_1 + r_2)^{n/r}|\lambda|^{-(1/r)}], \tag{5}$$

where $r = nk$, $k \geqq 2$.

$$\left| z - \frac{\delta_k b}{\delta_k - 1} \right| \leqq \frac{m + |\delta_k|(r_2 + 1)}{|\delta_k - 1|}, \qquad k = 1, \cdots, n, \tag{6}$$

where $n > r$, $w_k^n = \lambda$, $\delta_k^n = \lambda/(1 - \lambda)$, $k = 1, \cdots, n$;

$$m = \max_{1 \leqq k \leqq n} \frac{1}{|1 - w_k|}(|a - w_k b| + r_1 + |w_k| r_2).$$

Proof of (4). Let

$$F_1(z) = (z - \alpha)^n - \lambda(z - \beta)^r,$$

$$G(z) \equiv z^r \cdot F_1\left(\frac{1}{z}\right) = z^{r-n}(1 - z\alpha)^n - \lambda(1 - \beta z)^r;$$

hence $G(z)$ can also be written in the form:

$$G(z) = (-\alpha)^n(z - \gamma)^r - \lambda(1 - \beta z)^r,$$

where γ ranges over a circle including 0 and the points $1/\alpha$. If $G(z_0) = 0$, then $z_0 = (\delta + \gamma)/(1 + \delta\beta)$, where $\delta = \lambda^{1/r}(-\alpha)^{-n/r}$. Any zero of $F_1(z)$ is thus of the form $(1 + \delta\beta)/(\delta + \gamma)$. Let $C(a,b)$ denote the circle $|z - a| \leqq b$. If $\alpha \in C(a, r_1)$, then

$$\frac{1}{\alpha} \in C\left(\frac{\bar{a}}{|a|^2 - r_1^2}, \frac{r_1}{|a|^2 - r_1^2}\right)$$

and

$$\gamma \in C\left(\frac{e^{-i\phi}}{2(|a| - r_1)}, \frac{1}{2(|a| - r_1)}\right), \quad \phi = \arg a.$$

Thus

$$|\gamma| \leqq (|a| - r_1)^{-1} < |\lambda|^{1/r}(r_1 + |a|)^{-n/r} \leqq |\delta|$$

by our assumption $d(|a| - r_1) - 1 > 0$.

Now

$$z_0 \in C\left(\frac{\beta d^2 - \bar{\gamma}}{d^2 - |\gamma|^2}, \frac{d|\beta\gamma - 1|}{d^2 - |\gamma|^2}\right),$$

where $d = |\lambda|^{1/r}(r_1 + |a|)^{-n/r}$. Taking into account the inequalities $|\gamma| \leqq (|a| - r_1)^{-1}$, $|\beta| \leqq |b| + r_2$, we arrive at (4) after a short calculation.

Proof of (5). From $F_1(z) = (z - \alpha)^n - \lambda(z - \beta)^r$ it follows that

$$z^r F_1\left(\frac{1}{z} + \beta\right) = -\lambda + z^{r-n}[1 + (\beta - \alpha)z]^n.$$

If $F_1(\zeta) = 0$, then

$$-\gamma z^k + z^{k-1} - \mu = 0, \tag{7}$$

with $\zeta = 1/z + \beta$, $\mu = (\lambda)^{1/n}$, $\gamma = \alpha - \beta$.

The left-hand side of (7) can be written in the form

$$\left(\frac{\gamma}{\mu} z^k + 1\right)(z^{k-1} - \mu) - \frac{\gamma}{\mu} z^{2k-1}.$$

It follows by Szegö's Theorem [3, p. 60] that

$$|z| \geqq \frac{1}{2}\operatorname{Min}\left[|\mu|^{1/k}|\gamma|^{-(1/k)}, |\mu|^{1/k-1}\right]$$

$$= \frac{1}{2}\operatorname{Min}\left[|\lambda|^{1/r-n}, |\lambda|^{1/r}|\beta - \alpha|^{-n/r}\right]$$

and $|\zeta - \beta| \leqq 2\{\operatorname{Min}[|\lambda|^{1/r-n}, |\lambda|^{1/r}|\beta - \alpha|^{-(n/r)}]\}^{-1}$, (5) follows easily.

It is worthwhile to remark that by the same manipulation we can also obtain a lower bound for the zeros of $f(z) - \lambda g(z)$ namely writing

$$-\gamma z^k + z^{k-1} - \mu z + \frac{\mu}{\gamma} = \left(-\mu z + \frac{\mu}{\gamma}\right)\left(z^{k-1}\frac{\gamma}{\mu} + 1\right).$$

It follows by the same theorem due to Szegö that all the zeros of $-\gamma z^k + z^{k-1} - \mu z$ are in $|z| \leqq 2\operatorname{Max}(1/|\gamma|, (\mu/\gamma)^{1/k-1})$. The final estimate is $|\zeta - \beta| \geqq \{2\operatorname{Max}[|\alpha - \beta|^{-1}, |\lambda|^{1/nk}|\alpha - \beta|^{-k}]\}^{-1}$. To obtain a meaningful result it is necessary to suppose that $\operatorname{Min}|\alpha - \beta| > 0$; then

$$|\zeta - b| \geqq \{2\operatorname{Max}[(|a - b| - (r_1 + r_2))^{-1},$$
$$|\lambda|^{1/r}(|a - b| - (r_1 + r_2))^{-1/k}]\}^{-1} - r_2.$$

Proof of (6). Write $F_1(z) = f_1(z) - \lambda g_1(z)$, $f_1(z) = (z-\alpha)^n - \lambda(z-\beta)^n$, $g_1(z) = (z-\beta)^r - (z-\beta)^n$.

The zeros of $f_1(z)$ are in the union of the circles

$$C\left(\frac{a - w_k b}{1 - w_k}, \frac{r_1 + |w_k| r_2}{|1 - w_k|}\right)$$

(see, e.g., [3, p. 57]); hence in $C(0, r)$.

The zeros of $g_1(z)$ are in $C(b, r_2 + 1)$. Since $f_1(z)$ and $g_1(z)$ are both of degree n we can use the result in [3] to obtain (6).

We conclude this discussion by proving some results about the location of part of the zeros of the polynomial $(z-\alpha)^n - \lambda(z-\beta)^r$.

Theorem 4. *At least n zeros of the polynomial* $(z-\alpha)^n - \lambda(z-\beta)^r$ *are in the circle*

$$|z-\alpha| \leqq \frac{n}{r-n}|\alpha-\beta| \quad \text{if } n < r \leqq 2n,$$

$$|z-\alpha| \leqq |\alpha-\beta| \quad \text{if } r \geqq 2n,$$

and at most n zeros of the above polynomial are in the circle

$$|z-\alpha| \leqq |\alpha-\beta| \quad \text{if } n < r \leqq 2n,$$

$$|z-\alpha| \leqq \frac{n}{r-n}|\alpha-\beta| \quad \text{if } r \geqq 2n,$$

for all complex λ.

Proof. By a straightforward calculation one obtains that $\mathrm{Re}((z-A)/(z-B)) > 0$ (<0) if and only if

$$z \notin C\left(\frac{A+B}{2}, \frac{|A-B|}{2}\right), \quad \left(z \in C\left(\frac{A+B}{2}, \frac{|A-B|}{2}\right)\right)$$

for $A \neq B$.

Now

$$\frac{\partial}{\partial\theta} \arg_{|z-\alpha|=R}\left[\frac{(z-\alpha)^n}{-\lambda(z-\beta)^r}\right] \tag{8}$$

$$= \mathrm{Re}\left[(z-\alpha)\left(\frac{n}{z-\alpha} - \frac{r}{z-\beta}\right)\right] = (n-r)\mathrm{Re}\left[\frac{z + \dfrac{r\alpha - n\beta}{n-r}}{z-\beta}\right].$$

Since $n < r$ it follows that (8) is positive if and only if

$$z \in C\left(\frac{r(\alpha+\beta)-2n\beta}{2(r-n)}, \frac{r(\alpha-\beta)}{2(r-n)}\right). \tag{9}$$

In this case

$$\begin{aligned}
&\underset{|z-\alpha|=R}{\Delta \arg}\left[(z-\alpha)^n-\lambda(z-\beta)^r\right] \\
&\qquad = \underset{|z-\alpha|=R}{\Delta \arg}\left[\frac{(z-\alpha)^n}{-\lambda(z-\beta)^r}+1\right] + \underset{|z-\alpha|=R}{\Delta \arg}\left[(-\lambda)(z-\beta)^r\right] \\
&\qquad \leqq \underset{|z-\alpha|=R}{\Delta \arg}\left[\frac{(z-\alpha)^n}{-\lambda(z-\beta)^r}\right] + \underset{|z-\alpha|=R}{\Delta \arg}\left[-\lambda(z-\beta)^r\right] = 2\pi n.
\end{aligned}$$

Thus if

$$C(\alpha, R) \subset C\left(\frac{r(\alpha+\beta)-2n\beta}{2(r-n)}, \frac{r|\alpha-\beta|}{2(r-n)}\right),$$

then the polynomial $(z-\alpha)^n-\lambda(z-\beta)^r$ has at most n zeros in the circle $C(\alpha, R)$. It is easy to see that we can take

$$R = \frac{r-|r-2n|}{2(r-n)}|\alpha-\beta|.$$

This proves the second part of the theorem. Similarly

$$\frac{\partial}{\partial\theta}\arg\left[\frac{(z-\alpha)^n}{-\lambda(z-\beta)^r}\right] < 0$$

if and only if

$$z \notin C\left(\frac{r(\alpha+\beta)-2n\beta}{2(r-n)}, \frac{r|\alpha-\beta|}{2(r-n)}\right),$$

and we can set

$$R = \frac{|\alpha-\beta|}{2(r-n)}(r+|r-2n|).$$

It follows in particular that for $r = 2n$, the circle $|z-\alpha| \leqq |\alpha-\beta|$ contains exactly n zeros of the polynomial $(z-\alpha)^n-\lambda(z-\beta)^r$.

The following theorem generalizes a result due to Biernacki and Jankowski [1], [2].

THEOREM 5. *Let* $P(z) = a_p z^p + a_{p-s}z^{p-s} + \cdots + a_0$, $Q(z) = b_q z^q + b_{q-t}z^{q-t} + \cdots + b_0$. $a_p b_q \neq 0$, $q > p$, $s \geqq 1$, $t \geqq 1$ *have all their zeros in the circles* $|z| \leqq R_1$ *and* $|z| \leqq R_2$, *respectively. Let* $r = \mathrm{Min}(s,t) \geqq 1$. *At least p zeros of the polynomial*

$$P(z) + \lambda Q(z)$$

are in the circle

$$|z| \leqq \operatorname{Max}\left\{ \left(\frac{qR_1^r + pR_2^r}{q-p}\right)^{1/r}, R_2 \right\}. \tag{10}$$

Proof.

$$\operatorname*{Max}_{|z|=R} \frac{d}{d\theta} \arg \frac{P(z)}{Q(z)} \leqq \operatorname*{Max}_{|z|=R} \frac{d}{d\theta} \arg P(z) - \operatorname*{Min}_{|z|=R} \frac{d}{d\theta} \arg Q(z).$$

For $R > \operatorname{Max}(R_1, R_2)$ we have:

$$\begin{aligned} \operatorname*{Max}_{|z|=R} \frac{d}{d\theta} \arg \frac{P(z)}{Q(z)} &\leqq \operatorname*{Max}_{|z|=R} \operatorname{Re} \sum_{k=1}^{p} \frac{z}{z-\alpha_k} - \operatorname{Min} \operatorname{Re} \sum_{k=1}^{q} \frac{z}{z-\beta_k} \\ &\leqq p \operatorname*{Max}_{|z|=R} \operatorname{Re} \frac{z}{z-\alpha} - q \operatorname*{Min}_{|z|=R} \operatorname{Re} \frac{z}{z-\beta} \qquad (11) \\ &\leqq p \frac{R}{R-|\alpha|} - q \frac{R}{R+|\beta|}, \end{aligned}$$

where α_k, β_k are the zeros of $P(z)$ and $Q(z)$, respectively, and the functions $\alpha(z), \beta(z)$ satisfy $|\alpha(z)| \leqq R_1^r/R^{r-1}$, $|\beta(z)| \leqq R_2^r/R^{r-1}$. This follows by a recent result due to Walsh [4]. If the m_k, α_k, and z are given with $m_k > 0$, $|\alpha_k| \leqq A$, $|z| > A$, and $\sum_{k=1}^{n} m_k \alpha_k^l = 0$ for $l = 1, 2, \cdots, j$, then $\alpha = \alpha(z)$ as defined by the equation

$$\pi_{k=1}^{n}(z-\alpha_k)^n = (z-\alpha)^n$$

satisfies the inequality

$$|\alpha(z)| \leqq A^{j+1}/|z|^j.$$

Under the same conditions except that now $|\alpha_k| \geqq A$, $|z| < A$, and $\sum_{k=1}^{n} m_k \alpha_k^{-l} = 0$ and $l = 1, 2, \cdots, j$, we have

$$|\alpha(z)| \geqq A^{j+1}/|z|^j.$$

In deriving (11) we also notice that

$$\operatorname{Re}\left(\frac{z}{z-\alpha}\right) \leqq \left|\frac{z}{z-\alpha}\right| \leqq \frac{R}{R-|\alpha|}$$

and

$$\operatorname{Re}\left(\frac{z}{z-\beta}\right) = \frac{R(R - r\cos(\theta-\phi))}{R^2 + r^2 - 2rR\cos(\theta-\phi)},$$

with $\beta = re^{i\theta}$, $z = \mathrm{Re}^{i\phi}$.

The last expression is an increasing function of $\cos(\theta - \phi)$ and attains its minimum for $\cos(\theta-\phi) = -1$. Hence $\operatorname{Re}(z/(z-\beta)) \geqq R/(R+|\beta|)$. It follows now from (11) that

$$\operatorname*{Max}_{|z|=R} \frac{d}{d\theta} \arg \frac{P(z)}{Q(z)} \leqq p \frac{R^r}{R^r - R_1^r} - q \frac{R^r}{R^r + R_2^r} < 0$$

for $R^r > ((pR_2^r + qR_1^r)/(q-p))$. It is enough to set

$$R = \operatorname{Max}\left[\left(\frac{pR_2^r + qR_1^r}{q-p} \right)^{1/r}, R_2 \right]$$

which implies $R \geqq \operatorname{Max}(R_1, R_2)$.

Now one proves similarly to what has been done in Theorem 4 that

$$\mathop{\Delta}_{|z|=R} \arg(P + \lambda Q) \geqq 2\pi p$$

which concludes the proof.

It is clear that

$$R \leqq R' = \operatorname{Max}\left[\frac{pR_2 + qR_1}{q-p}, R_2 \right].$$

The estimate $|z| \leqq R'$ is due to Biernacki [1]. For large r, R tends to $\operatorname{Max}(R_1, R_2)$.

Bibliography

1. M. Biernacki, *Sur les équations algébriques contenant des paramètres arbitraires*, Bull. Acad. Polon. Sci. Sér. A (1927), 541-685.

2. W. Jankowski, *Sur les zéros d'un polynomial contenant un paramètre arbitraire*, Ann. Polon. Math. **3** (1957), 304-311.

3. M. Marden, *The geometry of the zeros of a polynomial in a complex variable*, Math. Surveys No. 3, Amer. Math. Soc., Providence, R. I. 1949.

4. J. L. Walsh, *A theorem of Grace on the zeros of polynomials, revisited*, Proc. Amer. Math. Soc. **15** (1964), 354-360.

5. Mishael Zedek, *Continuity and location of zeros of linear combinations of polynomials*, Proc. Amer. Math. Soc. **16** (1965), 78-84.

Harvard University,
Cambridge, Massachusetts